Perturbation theory for kinks and its application for multisoliton interactions in hydrodynamics

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Using an integrable Gardner equation as an example, a perturbation theory is developed for systems in which limiting-amplitude solitons exist in the form of a pair of distanced kinks. Approximate equations describing multisoliton interactions are derived and further used for modeling the evolution of an arbitrary set of solitons. The results are compared with an exact solution and numerical results. The theory is applied to data from observation of a train of strongly nonlinear internal waves in the ocean.

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I. INTRODUCTION

Interaction of solitary waves is now a classical problem being solved by exact methods such as the inverse scattering method (see, e.g., [1]) and by means of perturbation theories (see [2,3] and references therein). The latter are applicable to both integrable and nonintegrable equations, and in some cases they can be preferred over exact methods even for integrable equations. Indeed, if a sequence of many solitons with an arbitrary amplitude distribution is given as an initial condition, the use of this condition to obtain the specific exact solution may be a difficult problem itself, and the corresponding results are often physically unclear.

To demonstrate the application of the theory to be discussed below, we consider an integrable equation, the so-called Gardner equation (also named combKdV, extended KdV, etc.) which is a generalization of the Korteweg–de Vries (KdV) equation containing both quadratic and cubic nonlinearities:

$$\Phi_t + 6(\Phi - \Phi^2)\Phi_x + \Phi_{xxx} = 0. \tag{1}$$

This equation has been considered in numerous papers (e.g., [4,5]). In particular, it has important applications in physical oceanography (see an example in Sec. III). The choice of an integrable equation will enable us to compare the approximate solutions with those following from exact solutions. At the same time, the suggested method is not based on integrability and can evidently be employed for nonintegrable systems.

The known feature of solitary solutions of this equation is the existence of a soliton with a limiting amplitude $A_s = 1$ such that when the amplitude approaches this quantity, the soliton becomes close to two separated fronts, or kinks, and the distance between them tends to infinity.

In this paper, multisoliton interactions in the framework of this equation are considered. An approximate approach to be used here stems from the direct perturbation method for solitons developed previously by two of the authors [6]. This method was used to describe the interaction of localized waves as a process similar to a collision of classical particles. However, a straightforward application of the corresponding scheme to multisoliton interactions in the framework of Eq. (1) proves to be insufficient for the following reasons. First, the longer solitons with amplitudes close to the limiting one can be deformed in the course of interaction. Second, for a set of multiple solitons, it is constructive to describe such "collective" parameters as soliton amplitudes or positions (phases) in terms of the space-time modulation of these parameters.

The theoretical approach suggested below represents a soliton as a compound of two kinks and uses the method of matched asymptotic expansions for kinks upon retaining, along with a "corpuscular" description of solitons as particles, some wave properties of the kink sequence—i.e., the finite velocity of the perturbation ("envelope") propagation in the course of interaction as suggested in [7]. This aspect is a similarity with Whitham's theory for nonlinear quasiperiodic waves with slowly varying parameters [8]. However, here we do not assume that the basic solution is periodic. The solutions obtained in such a way are compared with the exact solutions and numerical results. Finally, general results are applied to oceanographic data describing a strongly nonlinear solitary internal wave train.

II. GENERAL SCHEME

The known solitary solutions to Eq. (1) can be represented as

$$\Phi_s(x,t) = \frac{k}{2}\left[\tanh\left(\frac{k}{2}(x - k^2 t + \Delta)\right) - \tanh\left(\frac{k}{2}(x - k^2 t - \Delta)\right)\right], \tag{2}$$

where $\Delta = k^{-1}\tanh^{-1}(k)$ and the parameter k lies in the range (0,1). At small k this solution coincides with the KdV soliton. At the other limit, when k is close to unity, the solution can be approximated as

$$\Phi_s(x,t,\varepsilon) \approx \frac{1-\varepsilon}{2}\left[\tanh\left(\frac{1-\varepsilon}{2}(x - k^2 t) + \frac{1}{4}\ln\frac{2}{\varepsilon}\right) - \tanh\left(\frac{1-\varepsilon}{2}(x - k^2 t) - \frac{1}{4}\ln\frac{2}{\varepsilon}\right)\right], \tag{3}$$

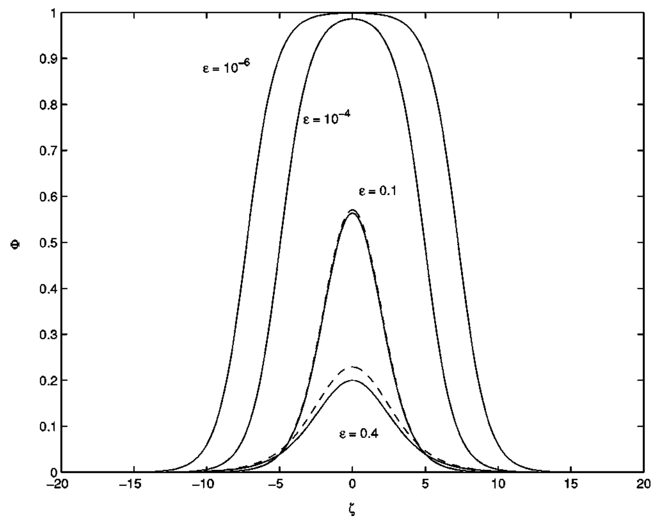

FIG. 1. Soliton profiles at four different values of the parameter $\varepsilon=1-k$: 10^{-6}, 10^{-4}, 0.1, and 0.4. Solid lines: exact solution. Dashed lines: approximate solution (the difference is noticeable only for the relatively small solitons with $1-k=0.1$ and 0.4).

where $\varepsilon=1-k$. As seen from Fig. 1, this expression provides a good approximation for the exact solution (2) even when the soliton is relatively far from the limiting one—up to, say, $\varepsilon=0.4$.

In the limit of $k=1$ the solution tends to a pair of infinitely distanced kinks, each propagating at the velocity $k^2=1$:

$$\Phi^{\pm}(x,t)=\frac{1}{2}\left[1\pm\tanh\frac{1}{2}(x-t)\right]. \quad (4)$$

Respectively, a strong soliton is close to a compound of two kinks of different polarities separated by an interval $2L$ large in comparison with the characteristic kink length:

$$\Phi_s(x,t)|_{k\to 1}\approx\Phi^{+}+\Phi^{-}-1=\frac{1}{2}\left[\tanh\left(\frac{1}{2}(x-t+L)\right)-\tanh\left(\frac{1}{2}(x-t-L)\right)\right]. \quad (5)$$

The latter expression is represented in a form characteristic of the method of matched asymptotic expansions: the solution consists of two regions of fast variation of the function Φ_s (in this case, kinks) separated by a region of slowly varying or almost constant solution (a plateau where $\Phi_s\approx 1$).

The form, Eq. (5), provides an incentive to represent a zero approximation for a general N-soliton solution, $\Phi_{Ns}^{(0)}$, as a sum of $2N$ noninteracting kinks of alternating signs:

$$\Phi_{Ns}^{(0)}(x,t)=\frac{1}{2}\sum_{i=1}^{2N}(-1)^{i+1}\tanh\frac{1}{2}[\zeta-S_i(\varepsilon t,\varepsilon x)], \quad (6)$$

where $\zeta=x-t$ and the phases S_i are slowly varying functions of x and t. Here ε is a small parameter of order $1-k$.

An algorithm for the construction of a general solution consists of finding local solutions in the vicinity of each kink and their subsequent matching. The full solution $\Phi_i(x,t)$ in the vicinity of an ith kink is represented as an asymptotic series in ε:

$$\Phi_i(x,t)=\Phi_i^{(0)}(\zeta-S_i)+\sum_{n=1}^{2N}\varepsilon^n\Phi_i^{(n)}(\zeta-S_i,\varepsilon t,\varepsilon x), \quad (7)$$

where $\Phi_i^{(0)}$ is a kink of the corresponding sign and each perturbation $\Phi^{(n)}$ is to be found as a function of $\zeta-S_i$ from a linear ordinary differential equation (ODE) that appears after substituting Eq. (7) into Eq. (2) and expanding in powers of ε:

$$\hat{L}_i\Phi_i^{(n)}=H_i^{(n)}(\Phi_i^{(0)},\ldots,\Phi_i^{(n-1)}). \quad (8)$$

Here

$$\hat{L}_i=\frac{d}{d\zeta}\left[-1+\frac{3}{2}\sec h^2\frac{1}{2}(\zeta-S_i)+\frac{d^2}{d\zeta^2}\right] \quad (9)$$

is a variational operator appearing after linearization of Eq. (1) at the ith kink and H_i are functionals obtained from lower-order approximations; in particular, $H_i^{(1)}$ depends on $\Phi_i^{(0)}$ and the derivatives of slowly varying phases S_i—namely,

$$H_i^{(1)}=\left(\frac{\partial S_i}{\partial t}+\frac{\partial S_i}{\partial x}\right)\frac{\partial\Phi_i^{(0)}}{\partial\xi}+2\frac{\partial S_i}{\partial x}\frac{\partial^3\Phi^{(0)}}{\partial\xi^3}. \quad (10)$$

Here and in what follows we omit the factor ε.

In the intervals between kinks the corresponding local perturbations must be matched. The solutions $\Phi_i=\varphi_i$ within these intervals are represented by a series analogous to Eq. (7) in which the main term $\varphi_i^{(0)}$ is equal to 1 for soliton tops and 0 for intervals between solitons. The perturbations $\varphi_i^{(n)}$ satisfy a system similar to Eq. (8):

$$\hat{L}_0\varphi_i^{(n)}=h_i^{(n)}(\varphi_i^{(0)}\cdots\varphi_i^{(n-1)}),\quad \hat{L}_0=\frac{d}{d\zeta}\left[-1+\frac{d^2}{d\zeta^2}\right]. \quad (11)$$

Here the functionals $h_i^{(n)}$ are represented in a form analogous to Eq. (10). It should be noted that $h_i^{(1)}=0$ because $\varphi_i^{(0)}$ are constants (either 0 or 1).

In the first approximation, it is necessary to match perturbations $\Phi_i^{(1)}$ at two sides of the kink with the respective perturbations $\Phi_{i-1}^{(1)}$ and $\Phi_{i+1}^{(1)}$. To this order, the exponentially decreasing asymptotics of neighboring zero-order kinks must be included in the points located far from the kink centers—namely,

$$\Phi_{i-1}^{(0)}[(\zeta-S_{i-1})\to\infty]=\begin{cases}e^{(-\zeta-S_{i-1})} & (\text{odd } i),\\ 1-e^{-(\zeta-S_{i-1})} & (\text{even } i),\end{cases} \quad (12)$$

and

$$\Phi_{i+1}^{(0)}[(\zeta-S_{i+1})\to-\infty]=\begin{cases}1-e^{\zeta-S_{i+1}} & (\text{odd } i),\\ e^{\zeta-S_{i+1}} & (\text{even } i).\end{cases} \tag{13}$$

These asymptotics are assumed to be of order ε in the vicinity of the neighboring ith kink—i.e., $\exp[-|S_{i\pm1}-S_i|]\sim\varepsilon$.

It is essential that the same exponents form the first-order solutions $\varphi_i^{(1)}$ between the kinks: since, as mentioned, $h_i^{(1)}=0$, Eq. (11) yields $\varphi_i^{(1)}=C_{i1}e^{\zeta}+C_{i2}e^{-\zeta}+C_{i3}$ with constant $C_{i1,2,3}$. It is constructive to explicitly single out the asymptotics given by Eqs. (12) and (13) in the first-order local solution:

$$\Phi_i^{(1)}=\tilde{\Phi}_i^{(1)}+(-1)^i(e^{+(\zeta-S_{i+1})}-e^{-(\zeta-S_{i-i})}). \tag{14}$$

Then we impose a condition of finiteness for all $\tilde{\Phi}_i^{(1)}$. Now the matching procedure is reduced to determination of coefficients $C_{i1,2,3}$, and a general solution $\Phi_{Ng}^{(1)}$ in the first approximation can be represented as a superposition of local solutions $\Phi_i^{(1)}$ minus their common asymptotics φ_i. As a result, exponential terms in $\Phi_i^{(1)}$ and $\varphi_i^{(1)}$ eliminate each other, and for N solitons we have

$$\Phi_{Ng}^{(1)}=\sum_{i=1}^{2N}(\tilde{\Phi}_i^{(1)}-\tilde{\Phi}_{i+}^{(1)}),$$

where

$$\tilde{\Phi}_{i+}^{(1)}=\tilde{\Phi}_i^{(1)}(\zeta\to\infty)=C_{3i}=\tilde{\Phi}_{(i+1)}^{(1)}(\zeta\to-\infty)=\tilde{\Phi}_{(i+1)-}^{(1)}. \tag{15}$$

As in other asymptotic perturbation schemes [3], the conditions of finiteness of $\tilde{\Phi}_i^{(1)}$ as solutions of Eq. (8) and, consequently, of the general solution $\Phi_{Ng}^{(1)}$ are equivalent to the following orthogonality (compatibility) conditions:

$$\int_{-\infty}^{\infty}\Phi_{i\zeta}^{(0)}d\zeta\int_0^{\zeta}\tilde{H}_i^{(1)}(\zeta,\tau,\rho)d\zeta'=0, \tag{16}$$

where

$$\tilde{H}_i^{(1)}=\left(\frac{\partial S_i}{\partial t}+\frac{\partial S_i}{\partial x}\right)\frac{\partial\Phi_i^{(0)}}{\partial\zeta}+2\,\frac{\partial S_i}{\partial x}\,\frac{\partial^3\Phi_i^{(0)}}{\partial\zeta^3} -(-1)^i\hat{L}_i(e^{-(\xi-S_{i-1})}-e^{(\xi-S_{i+1})}). \tag{17}$$

After substitution of $\Phi_i^{(0)}$ and integration, we obtain the equations for the kink phases:

$$\left(\frac{\partial S_i}{\partial t}+\frac{\partial S_i}{\partial x}\right)=-4[e^{-(S_{i+1}-S_i)}-e^{(S_{i-1}-S_i)}]. \tag{18}$$

The perturbation, Eq. (15), is limited at all ζ, but it is not uniformly valid yet; namely, the ratio $\Phi_{Ng}^{(1)}/\Phi_{Ng}^{(0)}$ is not small in areas between solitons where the zero-order function is itself small.

At this point, the explicit form of the perturbation, Eq. (15), satisfying Eqs. (8) and (9), can be represented as [7]

$$\Phi^{(1)}=\frac{1}{4}\sum_{i=1}^{2N}(-1)^{i+1}\left[\frac{dS_i}{dt}\tanh\left(\frac{\xi-S_i}{2}\right)+\left(\frac{\partial S_i}{\partial t}+3\,\frac{\partial S_i}{\partial x}\right)\times\left(\frac{\xi-S_i}{2}\right)\cosh^{-2}\left(\frac{\xi-S_i}{2}\right)\right], \tag{19}$$

where $d/dt=\partial/\partial t+\partial/\partial x$ is the derivative along the kink trajectory. At large $|\xi|$ the ratio of the last term in square brackets to the zero-order solution increases in proportion to $|\xi|$. To eliminate this peculiarity, the following condition should evidently be met:

$$\frac{\partial S_i}{\partial t}+3\,\frac{\partial S_i}{\partial x}=0,\quad i=1,2,...,2N. \tag{20}$$

Finally, an N-soliton solution taking interactions into account can be represented as

$$\Phi_{Ns}^{(0)}+\Phi_{Ns}^{(1)}=\frac{1}{2}\sum_{i=1}^{2N}(-1)^{i+1}\left(1-\frac{\partial S_i}{\partial x}\right)\tanh\frac{1}{2}[\zeta-S_i(x,t)]. \tag{21}$$

The slowly varying coefficients $(1-\partial S_i/\partial x)/2$ in this sum are x derivatives of the full phases of the kinks $[(\zeta-S_i)/2]$, just as the factor $k/2$ in the stationary soliton, Eq. (2), is the x derivative of the arguments of hyperbolic tangents. This means that the expression (21) represents an N-soliton solution as a superposition of quasistationary solitons.

According to Eq. (20), in this approximation the variations of kink phases propagate as stationary "envelope waves," $S_i(x,t)=S_i(\eta=x-3t)$, with a velocity 3 times the velocity of "carrier" stationary solitons of a limiting amplitude. By analogy with quasiharmonic waves, the velocity $v_g=d\omega/dk_{K=1}=3$, where $\omega(k)=k^3$ plays the role of the dispersion equation for a soliton, Eq. (2), can be interpreted as the group velocity for a set of solitons.

As a result, the partial-derivative equations (18) are reduced to the ordinary-difference, Katz–van Moerbeke equations

$$\frac{dS_i}{d\eta}=2(e^{-(S_{i+1}-S_i)}-e^{-(S_i-S_{i-1})}), \tag{22}$$

which realize the Bäcklund transform for Toda equations [9]. After differentiating, this system transforms to two Toda lattices:

$$\frac{d^2S_i}{d\eta^2}=4(e^{-(S_{i+2}-S_i)}-e^{-(S_i-S_{i-2})}). \tag{23}$$

Equation (23) describes two independent sequences (subsystems), one for even-numbered kinks and another for odd-numbered ones, which can be interpreted as frontal and trailing edges of solitons, respectively. Equation (22) serves as coupling conditions for these two subsystems. According to Eq. (22), in our case the Bäcklund transform is degenerated (the transform parameter is zero) and, consequently, the solutions to the subsystems (22) have a similar structure, possibly differing only in their parameter values (see below).

Let us now compare the approximate solution with the exact one. It is remarkable that the exact N-soliton solutions to the Gardner equation can be represented [7] in a form similar to Eq. (21) upon replacement of the approximate phases $S_i(\eta)$ by the exact ones, $-\theta_i(x,t)$. The functions $e^{\theta_i(x,t)}$ are found as the roots $z_p^{(\pm)}$ $(p=1,2,\ldots,N)$ of two algebraic equations $\Sigma_{i=0}^{N} a_p^{(\pm)} z^p = 0$ for kinks with odd $(\theta_i = \theta_p^{(+)},\ i=2p+1)$ and even $(\theta_i=\theta_p^{(-)},\ i=2p)$ numbers of i, respectively. The coefficients $a_p^{(\pm)}$ are related to the roots $z_p^{(\pm)}$ by the Vieta's theorem and have the following forms:

$$a_N^{(\pm)}=1,$$

$$-a_{N-1}^{(\pm)}=\sum_{p=1} \exp(-\theta_p^{(\pm)})=\sum_{p=1}^{N} \exp(\eta_p \pm k_p\Delta_p),$$

$$a_{N-2}^{(\pm)}=\sum_{1\leqslant p<p_2}^{N} \exp[-(\theta_{p_1}^{(\pm)}+\theta_{p_2}^{(\pm)})]$$

$$=\sum_{1\leqslant p<p_2}^{N} \exp(\eta_{p_1}\pm k_{p_1}\Delta_{p_1}+\eta_{p_2}\pm k_{p_2}\Delta_{p_2}+A_{p_1p_2})$$

$$\cdots$$

$$(-1)^N a_0^{(\pm)}=\exp\left(-\sum_{p=1}^{N} \theta_p^{(\pm)}\right)=\exp\left(\sum_{p=1}^{N} (\eta_p\pm k_p\Delta_p) + \sum_{1\leqslant p_1<p}^{N} A_{p_1p}\right). \tag{24}$$

Here $\eta_p=\varepsilon_p(x-3t)-\varepsilon_p^2(3-\varepsilon_p)t+\eta_{p0}$, $k_p\Delta_p=\tanh^{-1}k_p$, $e^{A_{pq}}=(k_p-k_q)^2/(k_p+k_q)^2$, and $\varepsilon_p=1-k_p$. The parameter k_p corresponds to the expression (2) for a pth soliton.

An important feature of this system in our case is that the phases θ_i are slowly varying functions of x and t. Indeed, their space-time scales are determined by the parameters ε_p^{-1}, which are large when k_p-1 is small and exceed unity for all possible values of k_p. In turn, the smallness of all ε_p and (similarly to what was supposed previously for $\exp[-|S_{i\pm1}-S_i|]$) of the functions $q_{ml}=\exp(\theta_l^{(\pm)}-\theta_m^{(\pm)})$ (here $m>l$) enables one to expand functions $a_n^{(\pm)}$ in a series

$$(-1)^n a_{N-n}^{(\pm)}=\exp\left(-\sum_{p=1}^{n} \theta_p^{(\pm)}\right)[1+O(q_{ml})] \tag{25}$$

and approximate the variables as $\eta_p=\varepsilon_p\eta+O(\varepsilon_p^2 t)$. Thus the expression (24) acquires a simpler form

$$a_N^{(\pm)}=1,$$

$$-a_{N-1}^{(\pm)}=e^{-\theta_1^{(\pm)}}=\sum_{p=1}^{N} e^{\varepsilon_p\eta+\eta_{0p}\pm\delta_p},$$

$$a_{N-2}^{(\pm)}=e^{-\theta_1^{(\pm)}-\theta_2^{(\pm)}}$$

$$=\sum_{1\leqslant p_1<p_2}^{N} e^{(\varepsilon_{p_1}+\varepsilon_{p_2})\eta+\eta_{0p_1}+\eta_{0p_2}\pm\delta_{p_1}\pm\delta_{p_2}+\tilde{A}_{p_1p_2}},$$

$$\cdots$$

$$(-1)^N a_0^{(\pm)}(\eta)=\exp\left(-\sum_{p=1}^{N} \theta_p^{(\pm)}\right)=\exp\left(\sum_{p=1}^{N} (\varepsilon_p\eta+\eta_{0p}\pm\delta p)+\sum_{1\leqslant p_1<p_2}^{N} \tilde{A}_{p_1p_2}\right). \tag{26}$$

Here $\delta_p=(1/2)\ln(2/\varepsilon_p)$ and $\exp(\tilde{A}_{p_1p_2})=[(\varepsilon_{p_1}-\varepsilon_{p_2})/2]^2$ are, respectively, the leading terms in expansions of the functions $k_p\Delta_p$ and $A_{p_1p_2}$ from Eq. (24) in powers of ε_p. As follows from Eq. (26), an explicit dependence of phases on the coefficients $a_{N-p}^{(\pm)}$ and, consequently, on η, has the form

$$\theta_p^{(\pm)}(\eta)=\ln\left[\frac{a_{N-p+1}^{(\pm)}(\eta)}{a_{N-p}^{(\pm)}(\eta)}\right],\quad \eta=x-3t. \tag{27}$$

It can be shown that the function (27) satisfies the Toda equation (23). This means that the approximate solution, Eq. (21), together with Eqs. (26) and (27), is equivalent to the first term in the corresponding expansion of the exact multisoliton solution. It is important that the exact solution for phases $\theta^{(\pm)}$ can be expanded in powers of the parameters ε_p and exponents q_{ml} for any $\varepsilon_p<1$ and $(\theta_m^{(\pm)}-\theta_l^{(\pm)})>0$—i.e., for arbitrary parameters of interacting solitons. This is true for all t except for those in which the neighboring solitons completely overlap.

III. TWO-SOLITON INTERACTION

A similarity between the general structures of the approximate and the exact solutions enables one to evaluate applicability of the approximate result by comparison of phase variables $S_i(x,t)$ and $\theta_i(x,t)$. Let us consider a two-soliton interaction as an example. As follows from Eq. (26), the exact solution for θ_i at $N=2$ reads (see also [4], where a two-soliton solution was found in a more cumbersome form)

$$\theta_{1,2}=\varepsilon_+\eta\mp\Delta_++\mu_+t-\cosh^{-1}\left[\frac{2-\varepsilon_+}{\varepsilon_-}\times\cosh(\varepsilon_-\eta\pm\Delta_-+\mu_-t)\right], \tag{28}$$

$$\theta_{3,4}=\varepsilon_+\eta\mp\Delta_++\mu_+t-\cosh^{-1}\left[\frac{2-\varepsilon_+}{\varepsilon_-}\times\cosh(\varepsilon_-\eta\pm\Delta_-+\mu_-t)\right]. \tag{29}$$

The corresponding approximate expressions for S_i follow from Eqs. (22) and (23) in the form

$$S_{1,2}=\varepsilon_+\eta\mp\delta_+-\ln\left[\frac{2}{\varepsilon_-}\cosh(\varepsilon_-\eta\pm\delta_+)\right], \tag{30}$$

$$S_{3,4}=\varepsilon_+\eta\mp\delta_++\ln\left[\frac{2}{\varepsilon_-}\cosh(\varepsilon_-\eta\pm\delta_+)\right]. \tag{31}$$

Here $\eta=x-3t$ and

$$2\varepsilon_\pm=\varepsilon_1\pm\varepsilon_2,\quad \varepsilon_{1,2}=1-k_{1,2},$$

$$2\mu_\pm=3(\varepsilon_1^2\pm\varepsilon_2^2)-(\varepsilon_1^3\pm\varepsilon_2^3),$$

$$2\Delta_\pm=\Delta_1\pm\Delta_2,\quad \Delta_{1,2}=\ln\frac{2-\varepsilon_{1,2}}{\varepsilon_{1,2}},$$

$$2\delta_\pm=\delta_1\pm\delta_2,\quad \delta_{1,2}=\ln\frac{2}{\varepsilon_{1,2}}.$$

The functions $\cosh^{-1}$ in Eqs. (28) and (29) can be expanded into a series uniformly converging at all $\varepsilon\in(0.1)$:

$$\cosh^{-1}(u_\pm)=\ln(2u_\pm)-\sum_{n=1}^{\infty}\frac{(2n-1)!!}{2n(2n)!!}u_\pm^{-2n}, \tag{32}$$

where

$$u_\pm=\frac{2-\varepsilon_+}{\varepsilon_-}\cosh(\varepsilon_-\eta\pm\Delta_-+\mu_-t).$$

It is easy to see that the phases S_i in Eqs. (30) and (31) are close to the main terms of these expansions, $\ln(2u_\pm)$; the difference is of order $\varepsilon_{1,2}^2$. Correspondingly, the constants $\delta_\pm$ and $2/\varepsilon_-$ in the approximate expressions coincide with the respective exact parameters $\Delta_\pm$ and $(2-\varepsilon_+)/\varepsilon_-$ up to the order of $O(\varepsilon_{1,2})$.

It is important that the absolute values of the main terms in these expansions, $\ln(2u_\pm)$, exceed the remaining sums for all values of x, t, and $\varepsilon\in(0.1)$. This enables one to successfully use the approximate solutions for interacting solitons even when their parameters k_1 and k_2 and, consequently, the amplitudes are not very close to unity. Due to the uniform validity in x and t, the approximate formulas can be used at all stages of interaction, even when the solitons are strongly overlapped and can not be visually identified.

Figure 2 shows a comparison between the exact and approximate solutions for the two-soliton interaction at $\varepsilon_1=10^{-6}$ and $\varepsilon_2=0.4$. It is seen that the approximate and exact solutions almost coincide at a point close to that of the maximal overlapping of solitons ($t=-1.5$), where the perturbation theory should, in principle, be inapplicable. At the same time, due to slightly different values of soliton velocities, initial distances between the solitons at which they approach the overlapping point differ for the two solutions. If the initial condition is given for well-separated solitons at the same distance, they would overlap in different points, but the shapes of these waves would be almost identical.

Note that the approximate solutions can be further improved by adding terms of order μ_+t in Eqs. (30) and (31),

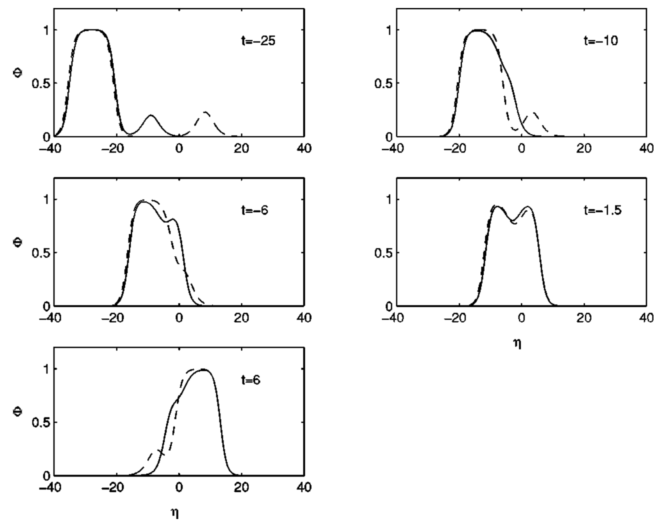

FIG. 2. Interaction of two solitons. Solid line: exact solution. Dashed line: approximate solution. Soliton parameters are $\varepsilon_1=10^{-6}$ and $\varepsilon_2=0.4$.

similarly to those in Eqs. (28) and (29). It can be shown that on that way the difference between the approximate and exact solutions, which is seen in Fig. 2, becomes even much smaller.

IV. ENVELOPE SOLITONS AND KINKS

Earlier it was shown [6] that in the framework of the direct perturbation method, the interaction of solitons in the KdV equation corresponds to that of two repulsive particles and that perturbation of a periodic set ("lattice") of such solitons can be described by the Toda system. As a result, these perturbations can propagate as envelope waves forming, in particular, Toda solitons and periodic envelope lattices; in turn, their perturbations also satisfy a Toda system, thus forming a "lattice hierarchy"—the same is true for, e.g., MKdV and sine-Gordon equations. According to the results of this paper, Gardner solitons are also repulsive. As already mentioned, the specifics of the processes considered here is that the soliton is treated as a compound of two kinks and, as a result, the acts of interaction for frontal and trailing kinks of each soliton are separated. This is especially important when a large number of solitons participate in the interaction.

Note first that Eq. (23) has two solutions in the form of infinite periodic trains (lattices) of solitons:

$$S_i=i\Lambda+V\eta+\begin{Bmatrix}0 & (\text{odd } i),\\ \delta & (\text{even } i).\end{Bmatrix} \tag{33}$$

These trains correspond to periodic solutions of the basic equation (1) having a period Λ and a characteristic soliton length δ (such as $-\Lambda<\delta<\Lambda$); as above, $\eta=x-3t$. Substituting these expressions into Eq. (22) yields a dispersion relation

$$3V=-2e^{-\Lambda}\cosh\delta, \tag{34}$$

which relates the velocity of a periodic wave, $1+3V$, in Eq. (1) to the parameters Λ and δ.

A general solution is described by the Toda equation (23) corresponding to the frontal and trailing kinks that are related by the Bäcklund transform, Eq. (22). In turn, these systems have solitary solutions which can be considered as pairs of kinks and are essentially the envelope solitons with respect to the solitons in the basic equation (2), modulating the sequences (33). A similar result has earlier been obtained for soliton lattices in the KdV equation [3]. However, in the present case, Eqs. (22) and (23) allow two different types of localized structures that correspond to the excitation of Toda solitons either in one or in both subsystems. The respective steady-state solutions are

$$S_i(\eta)=i\Lambda+V\eta+\begin{cases}0 & (\text{odd } i),\\ \ln\dfrac{\cosh[\lambda(i-2)-\beta\eta]}{\cosh[\lambda i-\beta\eta]} & (\text{even } i),\end{cases} \tag{35}$$

where $3V=-2e^{-\Lambda}\cosh 2\lambda$ and $\beta=e^{-\Lambda}\sinh 2\lambda$, and

$$S_i(\eta)=i\Lambda+V\eta+\begin{cases}\ln\dfrac{\cosh[\lambda(i-2)-\beta\eta]}{\cosh[\lambda i-\beta\eta]} & (\text{odd } i),\\ \delta+\ln\dfrac{\cosh[\lambda(i-2-\gamma)-\beta\eta]}{\cosh[\lambda(i-\gamma)-\beta\eta]} & (\text{even } i),\end{cases} \tag{36}$$

where $3V=-2e^{-\Lambda}\cosh 2\lambda$, $\beta=e^{-\Lambda}\sinh 2\lambda$, and $\sinh^2[\lambda(1-\gamma)]=e^{2\delta}\sinh[\lambda^2(1+\gamma)]$. Due to the symmetry of the subsystems, odd and even numbers for i in these expressions can be transposed. Note that in the first case, $\delta=\pm 2\lambda$, whereas in the second, δ and γ are independent parameters.

In the first of these envelope waves, Eq. (35), the kinks in one of the subsystems (frontal or trailing kinks) are eventually shifted at the distance of 4λ while the second subsystem remains unperturbed. These waves represent a sort of "exchange:" after the transition, the initial soliton widths become the distances between them and vice versa. These modulation waves may be called "envelope kinks."

In the waves described by Eq. (36) both subsystems of kinks are shifted at the same distance of 4λ. In this case wave profiles in the initial and final states are the same. Such waves can naturally be called envelope solitons.

Envelope solitons and kinks can interact in the same way as it occurs in the basic Gardner equation. Moreover, due to their exponential asymptotics, approximate equations describing these interactions must again be Toda systems (22) and (23), which have solutions in the form of "second-order" solitons and kinks. Continuing this process, a "hierarchy" of multiperiodic envelope waves can be constructed. It should be accentuated that the specifics of the considered solitons as compounds of kinks are reproduced at each level of such hierarchy.

V. MODELING OF THE EVOLUTION OF INTENSE INTERNAL WAVES BY A SEQUENCE OF SOLITONS

The Gardner equation (1) has important applications, in particular in physical oceanography for a description of nonlinear internal gravity waves generated by tides in coastal areas of the ocean. Such waves often have a form close to that of a group of solitons or a "solibore" (e.g., [15]). A similar equation, with an addition of higher-order dispersive terms, was first used for this purpose by Lee and Beardsley [10] and subsequently by other authors [4,11]. In some cases it gives a better description of moderately nonlinear internal wave forms than does the Korteveg–de Vries equation. Typically, however, not enough observational data are provided in publications to follow the evolution of a soliton group upon propagation at large distances. One of the few exceptions is the Coastal Ocean Probing Experiment (COPE), which was performed in 1995 on the northwestern shelf of the United States, off the coast of Oregon. Very strong, tide-generated internal waves were observed there in the form of a train of solitary waves [5,12,13]. In this case the solitons were so strong that no expansion of nonlinear terms leading to Eq. (1) is, strictly speaking, applicable. However, as shown in [5], Eq. (1), considered as a phenomenological model, describes such strong solitons reasonably well because for strong solitons it predicts wider solitons than the KdV equation does, and the width of their solitary solution only slightly varies in a significant range of amplitudes; both these features are consistent with the observational results.

In the case considered, the water stratification contains a sharp vertical variation of density, a pycnocline (see Fig. 2 of [5]), and its representation in the form of a two-layer model gives a good approximation for the real density profile. Hence we further consider nonlinear internal gravity waves in a two-layer fluid. In physical variables an equation similar to Eq. (1) reads (see, e.g., [5])

$$\frac{\partial\eta}{\partial t}+c\frac{\partial\eta}{\partial x}+\alpha\eta\frac{\partial\eta}{\partial x}-\alpha_1\eta^2\frac{\partial\eta}{\partial x}+\beta\frac{\partial^3\eta}{\partial x^3}=0. \tag{37}$$

Here η is the deviation from equilibrium of the liquid interface, $c=\sqrt{g(\Delta\rho/\rho)h_1h_2/(h_1+h_2)}$ is the propagation velocity of a long wave in the linear approximation, $\alpha=\frac{3}{2}c(h_2-h_1)/h_1h_2$ and $\alpha_1=(3c/8h_1^2h_2^2)[h_1^2+h_2^2+6h_1h_2]$ are the coefficients of nonlinearity, $\beta=c\ h_1h_2/6$ is the dispersion coefficient, h_1, h_2 are the thicknesses of the upper and lower liquid layers, respectively, and $\Delta\rho/\rho$ is the relative difference of liquid densities.

By substituting the variables

$$\eta=\left(\frac{\alpha}{\alpha_1}\right)\Phi,\quad t=t'\frac{\beta^{1/2}(6\alpha_1)^{3/2}}{\alpha^3},\quad x=ct+x'\frac{(6\alpha_1\beta)^{1/2}}{\alpha}, \tag{38}$$

into Eq. (37), we obtain Eq. (1).

From observations, we accept the following governing parameters for the two-layer model: $h_1=7$ m, $h_2=143$ m, and $\Delta\rho/\rho=3\times10^{-3}$. Then we obtain $\alpha=9.82\times10^{-2}\ \text{s}^{-1}$, $\alpha_1=0.35\times10^{-2}\ (\text{m s})^{-1}$, $\beta=76\ \text{m}^3/\text{s}$, and $c=0.458$ m/s; hence, for the maximal soliton in dimensional variables,

$\eta_{max}=\alpha/\alpha_1=28$ m and $V_{cr}=c+\alpha^2/6\alpha_1=2c$. Maximal observed soliton amplitudes (depressions of pycnocline) reached 25–27 m; i.e., in the framework of Eq. (37), these amplitudes are close to critical.[1] For modeling of an internal wave evolution, the records were taken from the COPE data for 26 September 1995 at two points: measurements at about 25 km from the shoreline [5] (site 1) and 20 km closer to the shoreline [13] (site 2). To apply Eq. (2), the time series at each of these points was recalculated to get the initial condition $\eta(x,t=0)$ by multiplying all time scales by the limiting soliton velocity $2c$.

According to the above theory, we approximated each solitary wave as a pair of kinks. To determine the initial coordinates of the kinks, we used the experimental records for impulse amplitudes and for time intervals between their maximums at site 1. The corresponding dimensionless distances $S_{i+1}-S_i$ between two kinks composing one and the same soliton (i.e., i is odd and $i+1$ even) were found according to the formula that follows from expression (2) with Δ determined from the measured amplitudes:

$$S_{i+1}-S_i=\ln\left(\frac{1+\alpha A_i/\alpha_1}{1-\alpha A_i/\alpha_1}\right), \tag{39}$$

where A_i is the dimensional amplitude of the ith soliton.

The distance between the neighboring kinks that belong to different solitons, $S_{i+1}-S_i$ at even i, is determined as a distance $\Delta x_{i+1,i}$ between the maxima of the corresponding impulses minus the half-sum of the distances, Eq. (39), determining the widths of these impulses. The distances $\Delta x_{i,i+1}$ are calculated from the measured time intervals $\Delta t_{i,i+1}$ between soliton maximums at site 1 as

$$\Delta x_{i+1,i}=V_{cr}\Delta t_{i+1,i}\,. \tag{40}$$

Here, as above, $V_{cr}=2c$. The resulting kink coordinates at site 1 were taken as the initial condition.

The system of first-order ODEs, Eq. (22), was solved numerically. We modeled the evolution of the groups of nine impulses (actually, the observed sequence was even longer, but beginning from the tenth impulse, the impulses are seated on a longer tidal depression pedestal and significantly overlapped). Calculation results were verified in two ways: first by direct computations of the basic equation (37) and then by comparison with the internal wave records at thermistor site 2 after passing 20 km toward the shore.

Figure 3 shows the initial sequence used for calculations according to the above theory and for numerical calculations from the basic equation. The dashed line corresponds to experimental observations [5,12] and solid line an approximation of the experimental data according to Eq. (2). Figure 4 shows the corresponding results of the calculation for the same soliton group at a distance of 20 km. In general, the approximate and numerical results are quite similar in both cases. In particular, the total duration of the group is practically the same, and the order in which the solitons follow each other differs only for the seventh and eighth impulses in Fig. 4(a), where they have already overlapped, unlike in Fig. 4(b). The differences are due to a small difference in soliton speeds (the same effect is seen in Fig. 2 for the two-soliton interaction), which can be considered as a measure of error in the approximate description. Anyway, for such a large distance (between 100 and 200 characteristic lengths of a soliton), the agreement can be considered to be quite satisfactory.

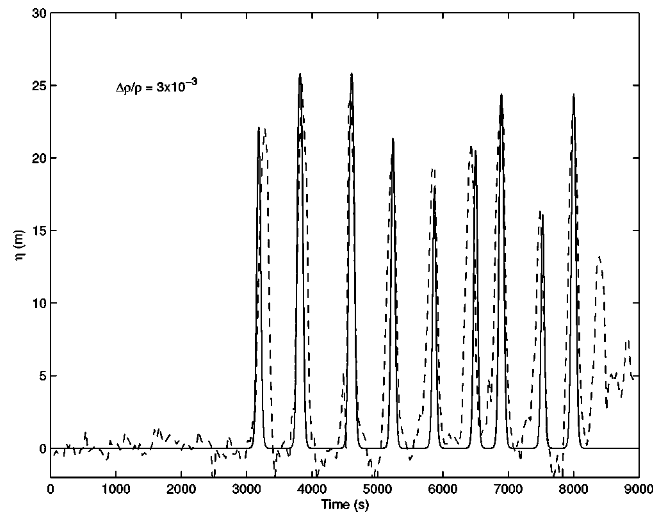

FIG. 3. Initial condition for the soliton group approximating a COPE experiment observation. Dashed line: experimental data. Solid line: their approximation by Eq. (2).

As seen from comparison with the initial state (Fig. 3), the group undergoes pronounced changes in the course of propagation: the relative positions of solitons have significantly changed. Also, the total duration of the soliton group has increased from about 5000 s to about 7000 s. Indeed, as known, interactions of soliton pairs always lead to energy flow from the rear soliton to the forward one and, as a result, to the acceleration of the latter (positive phase shift); this results in an effective stretching of the group.

Figure 5 gives the idea of observational results for site 2. It was plotted by using an isotherm restored from the thermistor chain data obtained by Trevorrow for the same soliton group in the same way as for the data included in [13]. Comparison with Fig. 4 shows major similarities in what regards the disposition of solitons on the time axis.

Note that this behavior is radically different from that in the KdV case, in which solitons tend to acquire the amplitudes linearly decreasing along the group. The main discrepancy between the theoretical model and the experiment (for both sites 1 and 2) is a an almost twofold difference in duration of impulses (2–3 min in experiment and 1–1.5 min in theory); also, the total group duration increases stronger than

[1]In direct calculations for the two-layer model (e.g., [16]) as well as in strongly nonlinear evolution equations [17], the limiting solitary wave amplitude also exists, but for a small density jump it is close to $(h_2-h_1)/2$. In our case it is about 68 m rather than 28 m in the Gardner model. However, for the amplitudes considered, the solitons are already significantly wider than those in the corresponding KdV equation, and the Gardner equation gives a good phenomenological description of them for amplitudes close to η_{max}.

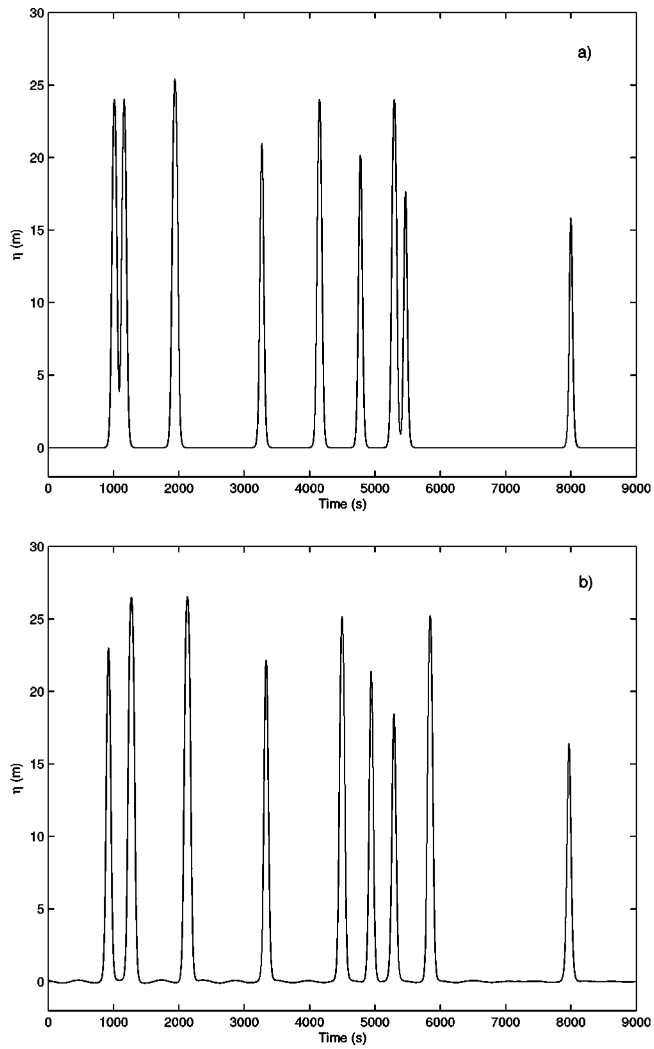

FIG. 4. Resulting distribution at a 20 km distance: (a) approximate solution and (b) numerical solution.

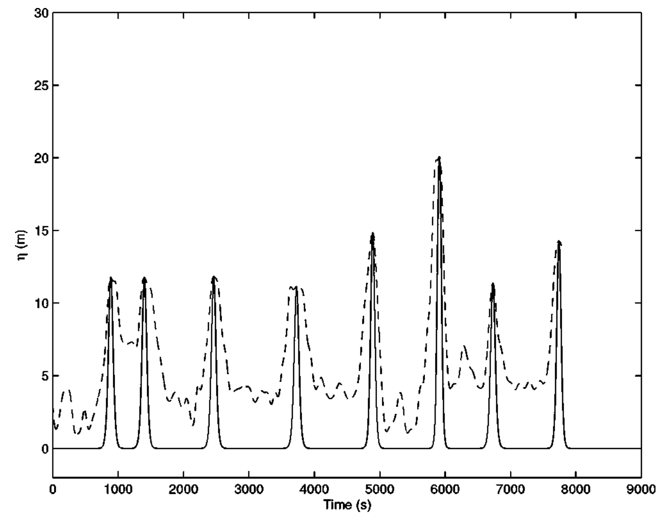

FIG. 5. Experimentally observed resulting distribution at a 20 km distance (dashed line) and its approximation by Eq. (2) (solid line).

in the theory. This is due to the limited applicability of the Gardner equation: in more realistic strongly nonlinear models [17] the solitons are broader and are supposed to interact longer, due to a smoother dependence of soliton velocities on their amplitudes. Note also that observational data for the site 2 were available for a deeper isotherm than for site 1; this, along with possible attenuation, is a possible cause of the decrease of the amplitudes of all impulses in the group in site 2 in comparison with site 1.

VI. CONCLUDING REMARKS

In this paper, a version of the asymptotic perturbation method for describing multisoliton interactions was suggested based on consideration of individual kinks and their subsequent matching. The method combines a description of individual solitons with their collective behavior. For the Gardner equation, the equations for kink interactions are reduced to two Toda systems in which the kink phases propagate at a "group velocity," 3 times the velocity of an individual soliton. Comparison with the exact two-soliton solution shows that exact and approximate equations have a similar structure, and their solutions coincide in the first approximation and remain close to each other in a wide range of parameters of the solitons. Also, such collective structures as envelope kinks and solitons have been described. The theory is further applied to multisoliton interactions to be compared with direct computations of the Gardner equation. These results are obtained for soliton groups modeling the real observational data on multisolitonic internal waves in the coastal ocean.

Finally, it should again be emphasized that the integrable Gardner equation was chosen in order to compare the approximate and exact solutions, whereas the method itself is irrelevant to the feature of integrability: it is based on soliton (kink) asymptotics that can be found from linearization of a stationary ODE describing the soliton in any model, as was demonstrated earlier by the direct perturbation method [6]. It is planned to apply this approach to nonintegrable equations (e.g., those developed recently for strongly nonlinear internal waves [14,17]) and to take into account other perturbing factors such as small losses and cylindrical divergence of wave fronts.

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